SOLUTION OF PROBLEMS IN NUMBER TWO.

SOLUTIONS of problems in number 2 have been received as follows:

From S. L. Curry, 151; M. B. W. Granger, 151; Henry Heaton, 160 and 161; Prof. E. W. Hyde, 156 and 160; Prof. A. Hall, 160; G. B. Halsted, 156; Chas. H. Kummell, 155 and answer to Prof. Hall's query; Prof. W. W. Johnson, 156 and 158; Christine Ladd, 157; Artemas Martin, 159 and 161; Dr. A. B. Nelson, 151 and 156; W. L. Marcy, 154, 158 and 159; J. B. Mott, 153; O. H. Merrill, 151; Prof. Orson Pratt, 154; Prof. J. Scheffer, 151, 153, 154, 158 and 159; E. B. Seitz, 159 and 161; C. T. Thompson, 151. Dr. A. B. Nelson, Alex. Evans, and T. P. Stowell, each answered Mr. Heal's query, and R. J. Adcock and W. Stille, Prof. Hall's.

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151. "What is the altitude of the maximum cylinder which can be inscribed in a given paraboloid?"

SOLUTION BY O. H. MERRILL, SOUTH RUTLAND, N. Y.

Let a be the altitude of the given parabola, p, its parameter, x the distance from its vertex to the upper extremity of the required cylinder and r the radius of the cylinder. Then we have

$r^2 = px$; $\therefore$ $\pi px(a - x) =$ vol. of cyl. $=$ a max., $\therefore$ $ax - x^2 =$ a max.

Putting the first differential coefficient equal to zero we get $x = \frac{1}{2}a$.

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152. "Find the surface of a right conoid with a circular base."

[No solution of this question has been received. — If we represent the required surface by S, we shall have

$$S = \int\!\!\int dx\,dy\left[1+\left(\frac{dz}{dx}\right)^2+\left(\frac{dz}{dy}\right)^2\right]^{\frac{1}{2}}.$$

For the equation of this surface see p. 126, Vol. II of this Journal.—Ed.]

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153. "Prove that if $(1 + 0\times\frac{1}{6}x\sqrt{-1})^{1\div 0} - (1 - 0\times\frac{1}{6}x\sqrt{-1})^{1\div 0} = \sqrt{-1}$ ... (1), $x = [(2\sqrt{-1})\div 0][(1 + \sqrt{-1})^0 - (1-\sqrt{-1})^0]$; ... (2) and find the real approximate value of x."

SOLUTION BY PROF. J. SCHEFFER, COLLEGE OF ST. JAMES, MD.

Developing $(1+y\times\frac{1}{6}x\sqrt{-1})^{\frac{1}{y}} - (1-y\times\frac{1}{6}x\sqrt{-1})^{\frac{1}{y}}$ by the Binomial Theorem and putting $y = 0$ in the resulting series, we get $2\sqrt{-1}\times\sin\frac{1}{6}x$. Hence we have, by (2), $2\sqrt{-1}\times\sin\frac{1}{6}x = \sqrt{-1}$; $\therefore$ $\sin\frac{1}{6}x = \frac{1}{2}$, $\therefore$ $x = \pi$.

Developing $(1+\sqrt{-1})^y-(1-\sqrt{-1})^y$ and putting $y=0$ in the resulting series we get $[(1+\sqrt{-1})^0-(1-\sqrt{-1})^0]\div 0=2\sqrt{-1}\times\frac{1}{4}\pi$. Consequently

$$x=\pi=[(2\sqrt{-1}\div 0)][(1+\sqrt{-1})^0-(1-\sqrt{-1})^0].$$

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154. "Find a general logarithmic theorem for the differentiation of

$$u=z^{x_1^{x_2^{\cdot^{\cdot^{\cdot^{x_n}}}}}}$$, z, x_1, x_2, &c, being any functions of one variable as x."

SOLUTION BY PROF. SCHEFFER.

Putting $u=z^{v_1}$; $v_1=x_1^{v_2}$; $v_2=x_2^{v_3}$; $v_3=x_3^{v_4}$, etc., we find, by taking natural logarithms, $\log u=v_1\log z$; $\log v_1=v_2\log x_1$; $\log v_2=v_3\log x_2$, etc. Differentiating each of these equations:

$$\frac{du}{u}=v_1\frac{dz}{z}+\log z.dv_1;\quad \frac{dv_1}{v_1}=v_2\frac{dx_1}{x_1}+\log x_1.dv_2;\quad \frac{dv_2}{v_2}=v_3\frac{dx_2}{x_2}+\log x_2.dv_3,$$

etc. Substituting the values of dv_1, dv_2, etc., we obtain

$$\frac{du}{u}=v_1\frac{dz}{z}+v_1v_2.\log z\frac{dx_1}{x_1}+v_1v_2v_3.\log z.\log x_1\frac{dx_2}{x_2}+v_1v_2v_3v_4.\log z.\log x_1\ldots$$

from which the law is apparent. We may put this expression in the form

$$\frac{du}{u}=v_1\left\{\frac{dz}{z}+v_2\log z\left[\frac{dx_1}{x_1}+v_3\log x_1\left(\frac{dx_2}{x_2}+v_4\log x_2\times\frac{dx_3}{x_3}+\ldots\right)\right]\right\}.$$

[The foregoing solution is substantially the same as the solutions sent by Prof. Pratt and Mr. Marcy, but we have adopted Prof. Scheffer's notation on account of its being the most convenient for the printer.]

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155. [A solution of this question by Chas. H. Kummell will be published in No. 4.]

156. "In a determinant of the nth degree the elements of the principal diagonal consist of units, and of the remaining elements those in the first column are each equal to a, those in the second column each equal to b and so on. Evaluate the determinant."

SOLUTION BY PROF. JOHNSON.

The determinant is a symmetrical function of the letters since any two of them may be interchanged by interchanging two rows and at the same time the corresponding columns. Its principal term is unity, and every other term contains at least two letters. Hence denoting the determinant by $\varDelta$ we may at once assume

$$\varDelta=1+A\Sigma ab+B\Sigma abc+\ldots.$$

Now if each of two of the letters equals unity, say if $k=l=1$ we shall have $\Delta=0$, since two columns will thus be rendered identical. Denoting by Σ' the sums taken for $n+2$ letters, k and l being excluded; we have when $k=l=1$, $\Sigma ab=\Sigma' ab+2\Sigma' a+1$, $\Sigma abc=\Sigma' abc+2\Sigma' ab+\Sigma' a$, &c. Hence putting $k=l=1$, the assumed equation becomes
$0=1+A+(2A+B)\Sigma' a+(A+2B+C)\Sigma' ab+(B+2C+D)\Sigma' abc+\ldots,$
which must hold independently of the values of a, b, c &c. Therefore equating to zero the coefficients we derive $A=1$, $B=2$, $C=-3$, &c., and we have

$$\Delta=1-\Sigma ab+2\Sigma abc-3\Sigma abcd+\ldots.$$

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157. "What is the entire number of double points which can be assumed arbitrarily on a curve of the nth degree?"

SOLUTION BY CHRISTINE LADD, UNION SPRINGS, N. Y.

A curve of the nth degree can be subjected to $\frac{1}{2}n(n+3)$ conditions. To require that a fixed point be a double point is equivalent to three conditions; hence the number of double points which can be assumed arbitrarily is never greater than $\frac{1}{6}n(n+3)$. In some cases it is not so great.

The number of double points possible to a proper n-ic is $\frac{1}{2}(n-1)(n-2)$. When this number is less than $\frac{1}{6}n(n+3)$, that is, when $n<6$, the curve cannot have so many as $\frac{1}{6}n(n+3)$ double points, but as many as it can have at all can be taken arbitrarily. When $n>5$, it is necessary to consider whether the assumption of $\frac{1}{6}n(n+3)$ double points will not cause the n-ic to break up into two curves of lower degree. For instance, if $n=6$, $\frac{1}{6}n(n+3)=9$, but through these nine points can be passed a determinate cubic, and, in general the only sextic having these points for double points is the cubic twice repeated. (Salmon, Higher Plane Curves, Art. 45.) But, in fact, 6 is the only value of n for which, in general, the curve can break up into two curves of lower degree, each passing through the $\frac{1}{6}n(n+3)$ doule points. That number of double points will consume all the conditions or all but two of them according as n is or is not a multiple of three. When n is odd, the n-ic may break up into a $\frac{1}{2}(n+1)$-ic and a $\frac{1}{2}(n-1)$-ic. But through a number of points by which a $\frac{1}{2}(n-1)$-ic can be determined, or through two more than that number, a $\frac{1}{2}(n+1)$-ic cannot pass, and by the number of points through which a $\frac{1}{2}(n+1)$-ic can pass, or by two less than that number, a $\frac{1}{2}(n-1)$-ic cannot be determined. Hence, when n is odd, the $\frac{1}{6}n(n+3)$ arbitrary double points cannot be the intersections of a $\frac{1}{2}(n-1)$-ic and a $\frac{1}{2}(n+1)$-ic. Still less can they be the intersections of any other carves into which the n-ic might break up. When n is even, we have to consider the danger of the n-ic breaking up into two $\frac{1}{2}n$-ics. The $\frac{1}{6}n(n+3)$ double points

are too many for the determination of a $\frac{1}{2}n$-ic when $n > 8$. When $n = 8$, we have fourteen arbitrary double points and two more conditions. If those conditions be that the curve have two more double points (not given), then two coincident quartics would meet the requirements, but if the two remaining conditions are not given then, in general, the 8-ic remains proper.

When $n = 6$, the number of double points which can be taken arbitrarily is eight. Dr. Salmon says (loc. cit.) that, *if the curve is required to have nine double points,* then not so many as eight of them can be assumed arbitrarily, but, in a letter which I have just received from him he says that this line will be removed from the next edition of his work.

To resume, the number of double points which can be taken arbitrarily when $n < 6$ is $\frac{1}{2}(n-1)(n-2)$, when $n = 6$ is $\frac{1}{6}n(n+3)-1$,
" $n = 7$ " $\frac{1}{6}n(n+3)$, " $n = 8$ " $\left\{\begin{array}{l}\frac{1}{6}n(n+3) \text{ or} \\ \frac{1}{6}n(n+3)-1\end{array}\right.$
as the curve is or is not required to have two more double points, and when

$$n > 8 \text{ it is } \tfrac{1}{6}n(n+3).$$

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158. "Let two concentric and similarly placed ellipses, infinitely near each other, be described, the semi-axes of the inner being a and b, and those of the outer $a + da$, and $b + da$; show that the minimum distance between their perimeters $= 2\sqrt{(ab)}da \div (a+b)$."

SOLUTION BY PROF. JOHNSON.

The perpendicular upon a tangent to the ellipse is

$$p = \sqrt{(a^2 \cos^2\alpha + b^2 \sin^2\alpha)}$$

where α is the inclination of p to the axis of x. The problem requires the minimum value of dp taken on the supposition that $da = db$, and α constant.

This is
$$dp = \frac{a \cos^2\alpha + b \sin^2\alpha}{\sqrt{(a^2\cos^2\alpha + b^2\sin^2\alpha)}}\, da.$$

Regarding $dp \div da$ as a function of α we find it to be a max. when $\sin\alpha \times \cos\alpha = 0$ and a min. when $\tan^2\alpha = a \div b$; substituting in the value of dp,

$$dp = \frac{2\sqrt{(ab)}}{a+b}\, da.$$

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159. "The first of two casks contains a gallons of wine" &c. [See p. 64.]

SOLUTION BY E. B. SEITZ.

Let u_n=the number of gallons of wine in the second cask after n oper'ns. Hence $a + c - u_n =$ the wine in the fiirst cask after the n operations, and $u_n + e(a+c-u_n) \div (a+b) =$ the wine in the second cask after putting e gals.

into it from the first in the $(n+1)$th operation, and in the second part of the operation $e\div(c+d+e)$ of the wine in the second cask is poured into the first; hence $(c+d)\div(c+d+e)$ of the wine remais.

$$\therefore \quad u_{n+1}-\left[\frac{(c+d)(a+b-e)}{(a+b)(c+d+e)}\right]u_n=e\left(\frac{a+c}{a+b}\right)\left(\frac{c+d}{c+d+e}\right). \quad . \quad . \quad (1)$$

The integral of (1) is

$$u_n = C\left[\frac{(c+d)(a+b-e)}{(a+b)(c+d+e)}\right]^n+\frac{(a+c)(c+d)}{a+b+c+d}. \quad . \quad . \quad . \quad (2)$$

When $n=0$, $u_0=c$; $\therefore C=c-[(a+c)(c+d)]\div(a+b+c+d)$.

$$\therefore u_n=c\left[\frac{(c+d)(a+b-e)}{(a+b)(c+d+e)}\right]^n+\frac{(a+c)(c+d)}{a+b+c+d}\left\{1-\left(\frac{(c+d)(a+b-e)}{(a+b)(c+d+e)}\right)^n\right\}.$$

160—161. [See page 64.]

SOLUTION BY HENRY HEATON.

160. Multiplying the given equation by $[y+f(x)]^{-n}$ we have

$$ydy[y+f(x)]^{-n}+(P-Qy)[y+f(x)]^{-n}dx.$$

If this is integrable

$$\frac{d}{dx}\left[\frac{y}{[y+f(x)]^n}\right]=\frac{d}{dy}\left[\frac{P-Qy}{[y+f(x)]^n}\right]; \therefore -ny\frac{df(x)}{dx}\Big(y+f(x)\Big)^{-n-1}$$

$$=-Q[y+f(x)]^{-n}-n(P+Qy)[y+f(x)]^{-n-1}.$$

$$\therefore \left(\frac{df(x)}{dx}+(n-1)Q\right)y-Qf(x)-nP=0. \quad \therefore f(x)=\frac{1-n}{n}\int Qdx,$$

and $P=-Qf(x)\div n$.

161. The centers of the random circles must be on the surface of a circle whose radius is r. Let O be the center of this circle, and let O' and O'' be the centers of the random circles. Put $OO'=x$ and $O'O''=y$.

If $y<(r-x)$, O'' must be in the circumference of a circle whose center is O' and whose radius is y; but if $y>(r-x)$, it will be in an arc of a circle, having the same center and radius, but terminated by the circumference of the circle O. The length of this arc is $2y\cos^{-1}[(x^2+y^2-r^2)\div(2xy)]$.

The area common to two circles whose radius is r and the distance between whose centers is y, is $A=2r^2\cos^{-1}(y\div 2r)-\frac{1}{2}y\sqrt{(4r^2-y^2)}$, and the av'ge area

$$=\frac{1}{r^4\pi^2}\int_0^r\left[\int_0^{r-x}A2\pi ydy+\int_{r-x}^{2r}A2y\cos^{-1}\left(\frac{x^2+y^2-r^2}{2xy}\right)dy\right]2\pi xdx$$

$$=\frac{2}{r^4\pi}\int_0^{2r}A^2ydy=r^2\pi-\frac{16r^2}{3\pi}.$$